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SPECTRAL RADIUS AND HAMILTONICITY OF GRAPHS WITH LARGE MINIMUM DEGREE

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Dedicated to the memory of Miroslav Fiedler

Abstract. Let G be a graph of order n and $\lambda(G)$ the spectral radius of its adjacency matrix. We extend some recent results on sufficient conditions for Hamiltonian paths and cycles in G. One of the main results of the paper is the following theorem:

Let $k \geqslant 2$, $n \geqslant k^3 + k + 4$, and let G be a graph of order n, with minimum degree $\delta(G) \geqslant k$. If

$$\lambda(G) \geqslant n - k - 1,$$

then G has a Hamiltonian cycle, unless $G = K_1 \vee (K_{n-k-1} + K_k)$ or $G = K_k \vee (K_{n-2k} + \bar{K}_k)$.

Keywords: Hamiltonian cycle; Hamiltonian path; minimum degree; spectral radius

MSC 2010: 05C50, 05C35

1. Introduction

This paper presents sufficient spectral conditions for Hamiltonian paths and cycles in graphs with large minimum degree.

Let $\lambda(G)$ denote the spectral radius of the adjacency matrix of a graph G. In 2010, Fiedler and Nikiforov in [9] gave some bounds on $\lambda(G)$ that imply the existence of Hamiltonian paths and cycles in G. This work motivated further research, as could be seen, e.g., in [1], [12], [13], [15], [14], [17], [20].

In the present paper we extend some recent results by Benediktovich [1], Li and Ning [13], and Ning and Ge [17]. To state those results we need to introduce three families of extremal graphs.

Write K_s and $\bar{K}_s$ for the complete and the edgeless graphs of order s. Given graphs G and H, write $G \vee H$ for their join and $G + H$ for their disjoint union.

First, for any $k \geqslant 1$ and $n \geqslant k+2$, let

$$L_k(n) := K_1 \vee (K_{n-k-1} + K_k).$$

That is to say, the graph $L_k(n)$ consists of a K_{n-k} and a K_{k+1} sharing a single vertex.

Second, for any $k \geqslant 1$ and $n \geqslant 2k+1$, let

$$M_k(n) := K_k \vee (K_{n-2k} + \bar{K}_k).$$

That is to say, the graph $M_k(n)$ consists of a K_{n-k} and a set of k independent vertices all joined to some k vertices from the K_{n-k}.

Finally, for any $k \geqslant 1$ and $n \geqslant 2k$, let

$$N_k(n) := K_k \vee (K_{n-2k-1} + \bar{K}_{k+1}).$$

That is to say, the graph $N_k(n)$ consists of a K_{n-k-1} and a set of $k+1$ independent vertices all joined to some k vertices from the K_{n-k-1}.

Note that for any admissible k and n, the graphs $L_k(n)$ and $M_k(n)$ contain no Hamiltonian cycles and the graph $N_k(n)$ contains no Hamiltonian paths, whereas the minimum degree of each of them is exactly k.

The graphs $M_k(n)$ and $N_k(n)$ were used by Erdős in [7] as extremal graphs in his results on Hamiltonicity of graphs with large minimum degree. Moreover, recently Li and Ning in [13] showed that $M_k(n)$ and $N_k(n)$ are also relevant for some spectral analogs of Erdős's results:

Theorem 1.1 (Li, Ning [13]). *Let $k \geqslant 0$ and let G be a graph of order n, with minimum degree $\delta(G) \geqslant k$.*

(1) *If $n \geqslant \max\{6k+10, (k^2+7k+8)/2\}$ and*

$$\lambda(G) \geqslant \lambda(N_k(n)),$$

then G has a Hamiltonian path, unless $G = N_k(n)$.

(2) *If $k \geqslant 1$, $n \geqslant \max\{6k+5, (k^2+6k+4)/2\}$, and*

$$\lambda(G) \geqslant \lambda(M_k(n)),$$

then G has a Hamiltonian cycle, unless $G = M_k(n)$.

Theorem 1.1 seems as good as one can get, and yet somewhat subtler and stronger statements have been proved for $k = 1, 2$.

Theorem 1.2 (Ning, Ge [17]). *Let $n \geqslant 7$ and let G be a graph of order n, with minimum degree $\delta(G) \geqslant 1$. If*

$$\lambda(G) > n - 3,$$

then G has a Hamiltonian path, unless $G = N_1(n)$.

Theorem 1.3 (Benediktovich [1]). *Let $n \geqslant 10$ and let G be a graph of order n, with minimum degree $\delta(G) \geqslant 2$. If*

$$\lambda(G) \geqslant n - 3,$$

then G has a Hamiltonian cycle, unless $G = L_2(n)$ or $G = M_2(n)$.

Note that in our renditions of Theorems 1.2 and 1.3, a few details have been omitted from the original statements in order to get more streamlined assertions. In this way one sees that these theorems are instances of more general statements, in which $\delta(G)$ is bounded by a parameter.

Thus, here we propose the following two theorems, which generalize Theorems 1.2 and 1.3, and strengthen Theorem 1.1 for n sufficiently large:

Theorem 1.4. *Let $k \geqslant 1$, $n \geqslant k^3 + k + 4$, and let G be a graph of order n, with minimum degree $\delta(G) \geqslant k$. If*

$$\lambda(G) \geqslant n - k - 1,$$

then G has a Hamiltonian cycle, unless $G = L_k(n)$ or $G = M_k(n)$.

Theorem 1.5. *Let $k \geqslant 1$, $n \geqslant k^3 + k^2 + 2k + 5$, and let G be a graph of order n, with minimum degree $\delta(G) \geqslant k$. If*

$$\lambda(G) \geqslant n - k - 2,$$

then G has a Hamiltonian path, unless $G = N_k(n)$ or $G = K_{n-k-1} + K_{k+1}$.

We shall give independent, self-contained proofs of Theorems 1.4 and 1.5, although many smart ideas could be readily borrowed from each of the papers [1], [13], and [17]. Crucial points of our arguments are based on the following straightforward theorems, whose proofs are nevertheless long and technical:

Theorem 1.6. *Let $k \geqslant 2$, $n \geqslant k^3 + k + 4$, and let G be a graph of order n, with minimum degree $\delta(G) \geqslant k$.*

(i) *If G is a subgraph of $L_k(n)$, then*

$$\lambda(G) < n - k - 1,$$

unless $G = L_k(n)$.

(ii) *If G is a subgraph of $M_k(n)$, then*

$$\lambda(G) < n - k - 1,$$

unless $G = M_k(n)$.

Theorem 1.7. *Let $k \geqslant 1$, $n \geqslant k^3 + k^2 + 2k + 5$, and let G be a graph of order n, with minimum degree $\delta(G) \geqslant k$.*

(i) *If G is a subgraph of $N_k(n)$, then*

$$\lambda(G) < n - k - 2,$$

unless $G = N_k(n)$.

(ii) *If G is a subgraph of $K_{n-k-1} + K_{k+1}$, then*

$$\lambda(G) < n - k - 2,$$

unless $G = K_{n-k-1} + K_{k+1}$.

The rest of the paper is structured as follows: In Section 2 we introduce some notation, recall some details about graph closure, and state a few results that will be used in the proofs of Theorems 1.4–1.7. The proofs themselves are given in Section 3. The last section is dedicated to a brief discussion and some open problems.

2. Notation and preliminaries

For graph notation and terminology undefined here we refer the reader to [2]. We write $A(G)$ for the adjacency matrix of a graph, and denote the quadratic form of $A(G)$ by $\langle A(G)\mathbf{x}, \mathbf{x}\rangle$, where $\mathbf{x}$ is a vector of size equal to the order of G. Note that if G is of order n and $\mathbf{x} = (x_1, \ldots, x_n)$, then

$$\langle A(G)\mathbf{x}, \mathbf{x}\rangle = 2 \sum_{\{i,j\} \in E(G)} x_i x_j.$$

If G is a graph of order n, we write $d_1, \ldots, d_n$ for the degrees of G indexed in ascending order.

A graph G is called *Hamiltonian-connected* if for any two vertices u and v of G, there is a Hamiltonian path in G whose ends are u and v.

We shall need the concept of graph closure, used implicitly by Ore in [19], [18], and developed further by Bondy and Chvátal in [3]: Fix an integer $k \geqslant 0$. Given a graph G, perform the following operation: if there are two nonadjacent vertices u and v with $d_G(u) + d_G(v) \geqslant k$, add the edge uv to $E(G)$. A *k-closure of G* is a graph obtained from G by successively applying this operation as long as possible. As it turns out, the k-closure of G is unique, that is to say, it does not depend on the order in which edges are added; see [3] for details.

Write $\mathrm{cl}_k(G)$ for the k-closure of G and note its main property:

If u and v are nonadjacent vertices of $\mathrm{cl}_k(G)$, then $d_{\mathrm{cl}_k(G)}(u) + d_{\mathrm{cl}_k(G)}(v) \leqslant k - 1$.

The usefulness of graph closure is demonstrated by the following facts, due essentially to Ore [19], [18]:

(A) A graph G of order n has a Hamiltonian path if and only if $\mathrm{cl}_{n-1}(G)$ has one.

(B) A graph G of order n has a Hamiltonian cycle if and only if $\mathrm{cl}_n(G)$ has one.

(C) A 2-connected graph G of order n is Hamiltonian-connected if and only if $\mathrm{cl}_{n+1}(G)$ is Hamiltonian-connected.

For convenience we restate the last two statements in a more usable form.

Theorem 2.1 (Ore [18]). *If G is a graph of order n and $d_u + d_v \geqslant n$ for any two distinct nonadjacent vertices u and v, then G has a Hamiltonian cycle.*

Theorem 2.2 (Ore [18]). *If G is a 2-connected graph of order n and $d_u + d_v \geqslant n+1$ for any two distinct nonadjacent vertices u and v, then G is Hamiltonian-connected.*

We shall also need two classical results of Chvátal [5] on Hamiltonicity of graphs.

Theorem 2.3 (Chvátal [5]). *Let G be a graph of order n, with degrees $d_1, \ldots, d_n$. If G has no Hamiltonian cycle, then there is an integer $s < n/2$ such that $d_s \leqslant s$ and $d_{n-s} \leqslant n - s - 1$.*

Corollary 2.4 (Chvátal [5]). *Let G be a graph of order n, with degrees $d_1, \ldots, d_n$. If G has no Hamiltonian path, then there is an integer $s < (n+1)/2$ such that $d_s \leqslant s - 1$ and $d_{n-s+1} \leqslant n - s - 1$.*

Finally, we shall need the following inequality, proved in [16]:

Corollary 2.5 ([10], [16]). *If G is a graph of order n, with m edges, and minimum degree δ, then*

$$\lambda(G) \leqslant \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}. \tag{2.1}$$

For connected graphs inequality (2.1) has been proved independently by Hong, Shu and Fang in [10].

The following observation is useful in applications of inequality (2.1).

Proposition 2.6 ([10], [16]). *If $2m \leqslant n(n-1)$, the function*

$$f(x) = \frac{x-1}{2} + \sqrt{2m - nx + \frac{(x+1)^2}{4}}$$

is decreasing in x for $x \leqslant n-1$.

3. Proofs

Proof of Theorem 1.6. Set for short $\lambda := \lambda(G)$, and let $\mathbf{x} = (x_1, \ldots, x_n)$ be a positive unit eigenvector to λ. Recall that Rayleigh's principle implies that

$$\lambda = \langle A(G)\mathbf{x}, \mathbf{x}\rangle.$$

The idea of the proofs below exploits the fact that both $M_k(n)$ and $L_k(n)$ consist of a K_{n-k}, together with an "outgrowth" of bounded order. It turns out that if n is large, all the edges incident to the "outgrowth" contribute to $\langle A(G)\mathbf{x}, \mathbf{x}\rangle$ much less than a single edge of the K_{n-k}.

Now we give the details.

Proof of (i). Assume that G is a proper subgraph of $M_k(n)$. Clearly, we may assume that G is obtained by omitting just one edge $\{u, v\}$ of $M_k(n)$.

Write X for the set of vertices of $M_k(n)$ of degree k, let Y be the set of their neighbors, and let Z be the set of the remaining $n-2k$ vertices of $M_k(n)$.

Since $\delta(G) \geqslant k$, we see that G must contain all the edges between X and Y. Therefore, $\{u, v\} \subset Y \cup Z$, with three possible cases: (a) $\{u, v\} \subset Y$; (b) $u \in Y, v \in Z$; (c) $\{u, v\} \subset Z$. We shall show that case (c) yields a graph of no smaller spectral radius than case (b), and that case (b) yields a graph of no smaller spectral radius than case (a).

Indeed, by symmetry, $x_i = x_j$ for any $i, j \in X$; likewise $x_i = x_j$ for any $i, j \in Y \setminus \{u, v\}$ and for any $i, j \in Z \setminus \{u, v\}$. Thus, let

$$\begin{aligned} x &:= x_i, \quad i \in X, \\ y &:= x_i, \quad i \in Y \setminus \{u, v\}, \\ z &:= x_i, \quad i \in Z \setminus \{u, v\}. \end{aligned}$$

Suppose that case (a) holds, that is, $\{u, v\} \subset Y$. Choose a vertex $w \in Z$, remove the edge $\{v, w\}$ and add the edge $\{u, v\}$. If $x_w > x_v$, swap the entries x_v and x_w; write $\mathbf{x}'$ for the resulting vector.

First, note that the obtained graph G' is covered by case (b) and that $\mathbf{x}'$ is a unit vector. We see that

$$\langle A(G')\mathbf{x}', \mathbf{x}'\rangle - \langle A(G)\mathbf{x}, \mathbf{x}\rangle = (x'_v - x_v) \sum_{i \in X} x_i \geqslant 0,$$

and, by the Rayleigh principle, $\lambda(G') \geqslant \lambda(G)$, as claimed.

Essentially the same argument proves that case (c) yields a graph of no smaller spectral radius than case (b). Therefore, we may assume that $\{u, v\} \subset Z$. Hence, the vertices u and v are symmetric, and so $x_u = x_v$. Set $t := x_u$ and note that the n eigenequations of G are reduced to four equations involving just the unknowns x, y, z, and t :

$$\lambda x = ky, \tag{3.1}$$

$$\lambda y = kx + (k-1)y + (n-2k-2)z + 2t, \tag{3.2}$$

$$\lambda z = ky + (n-2k-3)z + 2t, \tag{3.3}$$

$$\lambda t = ky + (n-2k-2)z. \tag{3.4}$$

We find that

$$x = \frac{k}{\lambda} y, \tag{3.5}$$

$$z = \Big(1 - \frac{k^2}{\lambda(\lambda+1)}\Big) y,$$

$$t = \frac{\lambda+1}{\lambda+2}\Big(1 - \frac{k^2}{\lambda(\lambda+1)}\Big) y. \tag{3.6}$$

Further, note that if we remove all edges between X and Y and add the edge $\{u, v\}$ to G, we obtain the graph $K_{n-k} + \bar{K}_k$. Letting $\mathbf{x}''$ be the restriction of $\mathbf{x}$ to K_{n-k}, we find that

$$\langle A(K_{n-k})\mathbf{x}'', \mathbf{x}''\rangle = \langle A(G)\mathbf{x}, \mathbf{x}\rangle + 2t^2 - 2k^2xy = \lambda + 2t^2 - 2k^2xy.$$

But since $\|\mathbf{x}''\| < 1$, we see that

$$\langle A(K_{n-k})\mathbf{x}'', \mathbf{x}''\rangle < \lambda(K_{n-k}) = n - k - 1,$$

that is,

$$\lambda + 2t^2 - 2k^2xy < n - k - 1. \tag{3.7}$$

Assume for a contradiction that $\lambda \geqslant n-k-1$. This assumption, together with (3.7) yields the inequality

$$k^2xy > t^2.$$

Now, (3.5) and (3.6) imply that

$$\frac{k^3}{\lambda}y^2 > \Big(\frac{\lambda+1}{\lambda+2}\Big)^2\Big(1 - \frac{k^2}{\lambda(\lambda+1)}\Big)^2 y^2.$$

Cancelling y^2 and applying Bernoulli's inequality to the right side, we get

$$\frac{k^3}{\lambda} > \Big(1 - \frac{2}{\lambda+2}\Big)\Big(1 - \frac{2k^2}{\lambda(\lambda+1)}\Big) > 1 - \frac{2}{\lambda+2} - \frac{2k^2}{\lambda(\lambda+1)}.$$

Referring to the inequalities $\lambda \geqslant n-k-1 \geqslant k^3+3$, we find that

$$k^3 > \lambda - \frac{2\lambda}{\lambda+2} - \frac{2k^2}{\lambda+1} > k^3 + 3 - 2 - \frac{2k^2}{k^3+4} > k^3,$$

a contradiction, completing the proof of (i). □

Proof of (ii). As in (i), assume that G is a subgraph of $L_k(n)$ obtained by omitting just one edge $\{u,v\}$ of $L_k(n)$. Recall that $L_k(n)$ consists of a K_{n-k} and a K_{k+1} sharing a single vertex, say w. Let Y be the set $\{w\}$, write X for the set of vertices of K_{k+1} that are distinct from w, and write Z for the set of vertices of K_{n-k} that are distinct from w.

Clearly, the condition $\delta(G) \geqslant k$ implies that $\{u,v\} \subset Y \cup Z$; among the three possible placements of $\{u,v\}$, the case $\{u,v\} \subset Z$ yields a graph with maximum spectral radius, so we assume that $\{u,v\} \subset Z$. Now, by symmetry, $x_i = x_j$ for any $i,j \in X$; likewise $x_u = x_v$ and $x_i = x_j$ for any $i,j \in Z \setminus \{u,v\}$. Thus, let

$$\begin{aligned} x &:= x_i, \quad && i \in X,\\ y &:= x_w, \quad && i \in Y,\\ z &:= x_i, \quad && i \in Z \setminus \{u,v\},\\ t &:= x_u = x_v. \end{aligned}$$

The n eigenequations of G now reduce to the four equations

$$\begin{aligned}\lambda x &= y + (k-1)x,\\ \lambda y &= kx + (n-2k-3)z + 2t,\\ \lambda z &= y + (n-2k-4)z + 2t,\\ \lambda t &= y + (n-2k-3)z.\end{aligned}$$

Hence, we find that

$$\begin{aligned}x &= \frac{1}{\lambda-k+1}y,\\ z &= \Big(1 - \frac{k}{(\lambda-k+1)(\lambda+1)}\Big)y,\\ t &= \frac{\lambda+1}{\lambda+2}\Big(1 - \frac{k}{(\lambda-k+1)(\lambda+1)}\Big)y.\end{aligned}$$

Further, if we delete all edges incident to vertices in X and add the edge $\{u, v\}$, we obtain the graph $K_{n-k} + \bar{K}_k$. Reasoning as in the proof of (i), we get the inequality

$$\frac{k(k-1)}{2(\lambda-k+1)^2}y^2 + \frac{k}{\lambda-k+1}y^2 > t^2 = \Big(\frac{\lambda+1}{\lambda+2}\Big)^2\Big(1 - \frac{k}{(\lambda-k+1)(\lambda+1)}\Big)^2 y^2,$$

which in turn yields

$$\frac{k(k-1)}{2(\lambda-k+1)} + k > \lambda - k + 1 - \frac{2(\lambda-k+1)}{\lambda+2} - \frac{2k}{\lambda+1}.$$

It is not hard to see that this inequality contradicts

$$\lambda \geqslant n-k-1 \geqslant k^3+3,$$

completing the proof of clause (ii) of Theorem 1.6. □

P r o o f of Theorem 1.7. The proof of (ii) is obvious. On the other hand, the proof of (i) is very similar to the proof of clause (i) of Theorem 1.6, so we shall skip it, stating just the starting system of equations, obtained with the same choice of variables as in equations (3.1)–(3.4):

$$\begin{aligned}\lambda x &= ky,\\ \lambda y &= (k+1)x + (k-1)y + (n-2k-3)z + 2t,\\ \lambda z &= ky + (n-2k-4)z + 2t,\\ \lambda t &= ky + (n-2k-3)z.\end{aligned}$$

Solving this system with respect to y and proceeding further as in the proof of clause (i) of Theorem 1.6, we get the inequality

$$\frac{k^2(k+1)}{\lambda} > \Bigl(1-\frac{2}{\lambda+2}\Bigr)\Bigl(1-\frac{2k(k+1)}{\lambda(\lambda+1)}\Bigr) > 1-\frac{2}{\lambda+2}-\frac{2k(k+1)}{\lambda(\lambda+1)},$$

which contradicts the inequalities $\lambda \geqslant n-k-2 \geqslant k^3+k^2+k+3$. $\square$

P r o o f of Theorem 1.4. Let $k \geqslant 1$, $n \geqslant k^3+k+4$ and let G be a graph of order n, with $\delta(G) \geqslant k$. Write m for the number of edges of G, and set $\delta := \delta(G)$.

Assume that $\lambda(G) \geqslant n-k-1$, but G has no Hamiltonian cycle. To prove the theorem we need to show that $G = L_k(n)$ or $G = M_k(n)$. Note that, in view of Theorem 1.6, it is sufficient to prove that $\mathrm{cl}_n(G) = L_k(n)$ or $\mathrm{cl}_n(G) = M_k(n)$, so this will be our main goal.

Clearly, $\mathrm{cl}_n(G)$ has no Hamiltonian cycle and

$$\delta(\mathrm{cl}_n(G)) \geqslant \delta(G) \geqslant k, \quad \lambda(\mathrm{cl}_n(G)) \geqslant \lambda(G) \geqslant n-k-1,$$

so for the rest of the proof we assume that $G = \mathrm{cl}_n(G)$. The main consequence of this assumption is that

$$d_i + d_j \leqslant n-1 \tag{3.8}$$

for every two nonadjacent vertices i and j.

Next, since G has no Hamiltonian cycle, Theorem 2.3 implies that there is an integer $s < n/2$ such that $d_s \leqslant s$ and $d_{n-s} \leqslant n-s-1$. Obviously, $s \geqslant \delta \geqslant k$, and we easily find an upper bound on $2m$:

$$\begin{aligned} 2m &= \sum_{i=1}^{s} d_i + \sum_{i=s+1}^{n-s} d_i + \sum_{i=n-s+1}^{n} d_i \\ &\leqslant s^2 + (n-2s)(n-s-1) + s(n-1) \\ &= n^2 - 2sn + 3s^2 + s - n. \end{aligned}$$

Clearly, the expression $n^2-2sn+3s^2+s-n$ is convex in s; hence it is maximal in s for $s = \delta$ or $s = (n-1)/2$. Hence, either

$$2m \leqslant n^2 - 2\delta n + 3\delta^2 + \delta - n \tag{3.9}$$

or

$$2m \leqslant n^2 - (n-1)n + 3\frac{(n-1)^2}{4} + \frac{n-1}{2} - n. \tag{3.10}$$

On the other hand, inequality (2.1) implies that

$$n-k-1 \leqslant \lambda(G) \leqslant \frac{\delta-1}{2}+\sqrt{2m-n\delta+\frac{(\delta+1)^2}{4}}.$$

Hence, in view of Proposition 2.6, we get

$$n-k-1 \leqslant \lambda(G) \leqslant \frac{k-1}{2}+\sqrt{2m-nk+\frac{(k+1)^2}{4}},$$

which, after some algebra, gives

$$2m \geqslant n^2-2kn+2k^2+k-n. \tag{3.11}$$

We shall now prove that $s=k$. Indeed, if $s \geqslant k+1$, then (3.9) and (3.10) imply that either

$$n^2-2(k+1)n+3(k+1)^2+k+1-n \geqslant n^2-2kn+2k^2+k-n$$

or

$$n^2-(n-1)n+\frac{3}{4}(n-1)^2+\frac{1}{2}(n-1)-n \geqslant n^2-2kn+2k^2+k-n.$$

Each of these inequalities leads to a contradiction, so we have $s=k$, and thus $\delta=k$. Therefore,

$$d_1=\ldots=d_k=k.$$

Our next goal is to show that $d_{k+1} \geqslant n-k-1-k^2$. Indeed, suppose that

$$d_{k+1} < n-k-1-k^2.$$

Now, using Theorem 2.3, we get

$$\begin{aligned}
2m &= \sum_{i=1}^{k} d_i + d_{k+1} + \sum_{i=k+2}^{n-k} d_i + \sum_{i=n-k+1}^{n} d_i \\
&< k^2+n-k-1-k^2+(n-2k-1)(n-k-1)+k(n-1) \\
&= n^2-2kn+2k^2+k-n,
\end{aligned}$$

contradicting (3.11). Hence $d_i \geqslant n-k-1-k^2$ for every $i \in \{k+1,\ldots,n\}$.

Next, we shall show that the vertices $k+1,\ldots,n$ induce a complete graph in G. Indeed, let $i \in \{k+1,\ldots,n\}$ and $j \in \{k+1,\ldots,n\}$ be two distinct vertices of G. If they are nonadjacent, then

$$\begin{aligned} d_i + d_j &\geqslant 2n - 2k - 2 - 2k^2 \\ &\geqslant n + k^3 + k + 4 - 2k - 2 - 2k^2 \\ &= (n-1) + k^3 - 2k^2 - k + 3 > n - 1, \end{aligned}$$

contradicting (3.8).

Write X for the vertex set $\{1, \dots, k\}$. Write Y for the set of vertices in $\{k+1, \dots, n\}$ having neighbors in X. It is easy to see that $Y \neq \emptyset$, since $|X| = k$ and so any vertex in X must have a neighbor in $\{k+1, \dots, n\}$.

In fact, every vertex from Y is adjacent to every vertex in X. Indeed, suppose that this is not the case, and let $w \in \{k+1, \dots, n\}$, $u \in X$, $v \in X$ be such that w is adjacent to u, but not to v. We see that

$$d_w + d_v \geqslant n - k + k = n,$$

contradicting (3.8).

Next, let $l := |Y|$ and note that $1 \leqslant l \leqslant k$, since $d_1 = k$. If $l = 1$, then $G = L_k(n)$, and if $l = k$, then $G = M_k(n)$. To finish the proof we shall show that if $1 < l < k$, then G has a Hamiltonian cycle, which contradicts the assumptions about G.

Indeed, let H be the graph induced by the set $X \cup Y$. Since $K_l \vee \bar{K}_k \subset H$ and $l \geqslant 2$, we see that H is 2-connected. Further, if u and v are distinct nonadjacent vertices of H, with degrees d'_u and d'_v, they must belong to X, and so $d'_u = d_u = k$ and $d'_v = d_v = k$. That is to say,

$$d'_u + d'_v = 2k > k + l.$$

By Theorem 2.2, H is Hamiltonian-connected, and it is easy to see that G has a Hamiltonian cycle.

The proof of Theorem 1.4 is completed. □

P r o o f of Theorem 1.5. Although this proof is very close to the proof of Theorem 1.4, we shall carry it in full, due to the numerous specific details.

Let $k \geqslant 1$, $n \geqslant k^3 + k^2 + 2k + 5$ and let G be a graph of order n, with $\delta(G) \geqslant k$. Write m for the number of edges of G, and set $\delta := \delta(G)$.

Assume that $\lambda(G) \geqslant n - k - 2$, but G has no Hamiltonian path. To prove the theorem we need to show that $G = N_k(n)$ or $G = K_{n-k-1} + K_{k+1}$. Note that, in view of Theorem 1.6, it is sufficient to prove that $\mathrm{cl}_n(G) = N_k(n)$ or $\mathrm{cl}_n(G) = K_{n-k-1} + K_{k+1}$, so this will be our main goal.

Clearly, $\mathrm{cl}_n(G)$ has no Hamiltonian path and

$$\delta(\mathrm{cl}_n(G)) \geqslant \delta(G) \geqslant k, \quad \lambda(\mathrm{cl}_n(G)) \geqslant \lambda(G) \geqslant n-k-2,$$

so for the rest of the proof we assume that $G = \mathrm{cl}_n(G)$. The main consequence of this assumption is that

$$d_i + d_j \leqslant n-2 \tag{3.12}$$

for every two nonadjacent vertices i and j.

Next, since G has no Hamiltonian path, Corollary 2.4 implies that there is an integer $s \leqslant n/2$ such that $d_s \leqslant s-1$ and $d_{n-s+1} \leqslant n-s$. Obviously, $s \geqslant \delta+1 \geqslant k+1$, and we easily find an upper bound on $2m$:

$$\begin{aligned} 2m &= \sum_{i=1}^{s} d_i + \sum_{i=s+1}^{n-s+1} d_i + \sum_{i=n-s+2}^{n} d_i \\ &\leqslant s(s-1) + (n-2s+1)(n-s-1) + (s-1)(n-1) \\ &= n^2 - 2sn + 3s^2 - s - n. \end{aligned}$$

Clearly, the expression $n^2 - 2sn + 3s^2 - s - n$ is convex in s; hence it is maximal in s for $s = \delta + 1$ or $s = n/2$. Therefore, either

$$2m \leqslant n^2 - 2(\delta+1)n + 3(\delta+1)^2 - (\delta+1) - n \tag{3.13}$$

or

$$2m \leqslant \frac{3}{4}n^2 - \frac{3}{2}n. \tag{3.14}$$

On the other hand, inequality (2.1) implies that

$$n-k-2 \leqslant \lambda(G) \leqslant \frac{\delta-1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}.$$

In view of Proposition 2.6, we get

$$n-k-2 \leqslant \lambda(G) \leqslant \frac{k-1}{2} + \sqrt{2m - nk + \frac{(k+1)^2}{4}},$$

which, after some algebra, gives

$$2m \geqslant n^2 - 2kn + 2k^2 + 4k - 3n + 2. \tag{3.15}$$

We shall prove that $s = k+1$. Indeed, if $s \geqslant k+2$, then (3.13) and (3.14) imply that either

$$n^2 - 2(k+2)n + 3(k+2)^2 - (k+2) - n \geqslant n^2 - 2kn + 2k^2 + 4k - 3n + 2$$

or

$$\frac{3}{4}n^2 - \frac{3}{2}n \geqslant n^2 - 2kn + 2k^2 + 4k - 3n + 2.$$

Each of these inequalities leads to a contradiction, so we have $s = k+1$ and thus $\delta = k$. Therefore,

$$d_1 = \ldots = d_{k+1} = k.$$

Our next goal is to show that $d_{k+2} \geqslant n - 2k - 2 - k^2$. Indeed, suppose that

$$d_{k+2} < n - 2k - 2 - k^2.$$

Now we get

$$\begin{aligned} 2m &= \sum_{i=1}^{k+1} d_i + d_{k+2} + \sum_{i=k+3}^{n-k} d_i + \sum_{i=n-k+1}^{n} d_i \\ &< k(k+1) + n - 2k - 2 - k^2 + (n-2k-2)(n-k-2) + k(n-1) \\ &= n^2 - 2kn + 2k^2 + 4k - 3n + 2, \end{aligned}$$

contradicting (3.15). Hence $d_i \geqslant n - 2k - 2 - k^2$ for every $i \in \{k+2, \ldots, n\}$.

We shall show that the vertices $k+2, \ldots, n$ induce a complete graph in G. Indeed, let $i \in \{k+2, \ldots, n\}$ and $j \in \{k+2, \ldots, n\}$ be two distinct vertices of G. If they are nonadjacent, then

$$\begin{aligned} d_i + d_j &\geqslant 2n - 4k - 4 - 2k^2 \\ &\geqslant n + k^3 + k^2 + 2k + 5 - 4k - 4 - 2k^2 \\ &= (n-2) + k^3 - k^2 - 2k + 3 > n - 2, \end{aligned}$$

contradicting (3.12).

Write X for the vertex set $\{1, \ldots, k+1\}$. Write Y for the set of vertices in $\{k+2, \ldots, n\}$ that have neighbors in X. If $Y = \emptyset$, then G is a disconnected graph and the order of its largest component is at most $n-k-1$. Also X induces a K_{k+1}. Clearly, the inequality $\lambda(G) \geqslant n-k-2$ implies that $G = K_{n-k-1} + K_{k+1}$, completing the proof if $Y = \emptyset$.

Now, suppose that $Y \neq \emptyset$. We shall show that every vertex in Y is adjacent to every vertex in X. Indeed, suppose that this is not the case, and let $w \in \{k+1, \ldots, n\}$, $u \in X$, $v \in X$ be such that w is adjacent to u, but not to v. We see that

$$d_w + d_v \geqslant n - k - 1 + k = n - 1,$$

contradicting (3.8).

Next, let $l := |Y|$ and note that $1 \leqslant l \leqslant k$, as $d_1 = k$. If $l = k$, then $G = N_k(n)$. To finish the proof we shall show that if $1 \leqslant l < k$, then G has a Hamiltonian path.

Indeed, let H be the graph induced by the set $X \cup Y$. Further, if u and v are distinct nonadjacent vertices of H, with degrees d'_u and d'_v, they must belong to X, and so $d'_u = d_u = k$ and $d'_v = d_v = k$. That is to say,

$$d'_u + d'_v = 2k \geqslant k + 1 + l.$$

By Theorem 2.1, H contains a Hamiltonian cycle, and hence G has a Hamiltonian path.

The proof of Theorem 1.5 is completed. $\square$

4. Concluding remarks

It should be noted that most of the results discussed in the present paper and in the references [1], [12], [13], [15], [14], [17], [20] deal exclusively with very dense graphs, which makes these results somewhat one-sided. Hoping to change this tendency, we would like to state two open problems.

First, recall that Dirac's theorem in [6] is probably the most famous sufficient condition for Hamiltonian cycles. Yet, no comparable spectral statement seems to be known so far.

Problem 4.1. Find a spectral sufficient condition for Hamiltonian cycles that would imply Dirac's sufficient condition.

Second, a deep result of Krivelevich and Sudakov in [11] establishes a sufficient condition on the second largest singular value of a regular graph that implies existence of Hamiltonian cycles. Two attempts have been made to extend this result to nonregular graphs, but these extensions forsake the adjacency matrix for other matrices ([4], [8]), so comparisons are difficult.

Hence, it is worth to reiterate the following problem, first raised in [11]:

Problem 4.2. Extend the result of Krivelevich and Sudakov to nonregular graphs, using the second largest singular value of the adjacency matrix.

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